

A geometric criterion to be pseudo-Anosov

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August 8, 2012

Abstract

If S is a hyperbolic surface and \mathring{S} the surface obtained from S by removing a point, the mapping class groups $\text{Mod}(S)$ and $\text{Mod}(\mathring{S})$ fit into a short exact sequence

$$1 \rightarrow \pi_1(S) \rightarrow \text{Mod}(\mathring{S}) \rightarrow \text{Mod}(S) \rightarrow 1.$$

We give a new criterion for mapping classes in the kernel to be pseudo-Anosov using the geometry of hyperbolic 3-manifolds. Namely, we show that if M is an ε -thick hyperbolic manifold homeomorphic to $S \times \mathbb{R}$, then an element of $\pi_1(M) \cong \pi_1(S)$ represents a pseudo-Anosov element of $\text{Mod}(\mathring{S})$ if its geodesic representative is “wide.” We establish similar criteria where M is replaced with a coarsely hyperbolic surface bundle coming from a δ -hyperbolic surface-group extension.

1 Introduction: mapping classes from fibrations

If X is a surface, let $\text{Mod}(X) = \pi_0(\text{Homeo}^+(X))$ be its mapping class group and let \mathring{X} be the surface obtained from X by removing a point.

Let N be a closed hyperbolic 3-manifold that fibers over the circle with fiber a surface S , and let $N_{\mathbb{Z}} \rightarrow N$ be the corresponding infinite cyclic covering of N . The long exact sequence of the fibration is concentrated in a short exact sequence

$$1 \longrightarrow \pi_1(S) \longrightarrow \pi_1(N) \longrightarrow \mathbb{Z} \longrightarrow 1 \tag{1.1}$$

which injects into the Birman exact sequence [4]

$$1 \longrightarrow \pi_1(S) \longrightarrow \text{Mod}(\mathring{S}) \longrightarrow \text{Mod}(S) \longrightarrow 1.$$

Choosing a lift t of the generator of \mathbb{Z} to $\pi_1(N)$, any element of $\pi_1(N)$ may be written uniquely as a product gt^k , where g is an element of $\pi_1(S)$. When k is nonzero, this element represents a pseudo-Anosov mapping class in $\text{Mod}(\mathring{S})$. When k is zero, this element lies in $\pi_1(S)$, and, by a theorem of Kra [14] (see also [12]), it is pseudo-Anosov in $\text{Mod}(\mathring{S})$ if and only if it fills S . These observations were first made by Ian Agol [2].

*The first author was supported in part by an NSF MSPRF and NSF grant DMS-1104871, the second author by NSF grants DMS-0603881 and DMS-0905748. Both authors were partially supported by the GEAR network.

Criterion 1 (Agol’s criterion). *A subgroup H of $\pi_1(N)$ is a purely pseudo-Anosov subgroup of $\text{Mod}(\mathring{S})$ if and only if every nontrivial element of $H \cap \pi_1(S)$ fills S .*

This topological criterion is very difficult to check. Our main theorem is a geometric criterion for an element of $\pi_1(N_{\mathbb{Z}})$ to be filling.

Theorem 5. *Let S be a closed oriented surface of euler characteristic $\chi = \chi(S) < 0$ and let ε and K be positive numbers. There is a $W = W(\chi, \varepsilon, K) > 0$ such that the following holds. Equip $M = S \times \mathbb{R}$ with any ε -thick hyperbolic structure, and let $\ell: M \rightarrow \mathbb{R}$ be a K -Lipschitz submersion. If Y is a proper incompressible subsurface of S and \mathcal{C}_Y is the convex core of the corresponding cover of M , then the width of \mathcal{C}_Y is at most W . In particular, if γ is a geodesic loop in M such that $\text{diam}(\ell(\gamma)) > W$, then γ fills S .*

If $\text{diam}(\ell(\gamma)) > W$, we say that γ is *wide*. Agol’s criterion then becomes:

Criterion 2 (Width criterion). *A subgroup H of $\pi_1(N)$ is a purely pseudo-Anosov subgroup of $\text{Mod}(\mathring{S})$ if every nontrivial element of $H \cap \pi_1(S)$ is wide.*

(Note that filling elements need not be wide.)

This criterion, and Theorem 5, arose out of the authors’ attempts to find purely pseudo-Anosov surface subgroups of mapping class groups by exploiting the abundance of surface subgroups of hyperbolic 3-manifold groups (see [11]).

In Section 3 we prove a generalization of Theorem 5 to the case of punctured surfaces, Theorem 9. The authors and S. Dowdall use these theorems to prove the following.

Theorem 3 (Dowdall–Kent–Leininger [7]). *Suppose N is a finite volume hyperbolic 3-manifold that fibers over the circle with fiber S and $G < \pi_1(N)$. As a subgroup of $\text{Mod}(\mathring{S})$, G is convex cocompact in the sense of Farb and Mosher [8] if and only if G is finitely generated and purely pseudo-Anosov.*

In particular, this answers a special case of Question 1.5 of [8], and generalizes Theorem 6.1 of [12].

In Section 4, we generalize Theorem 5 in a different direction by replacing M with a hyperbolic surface-group extension Γ .

Theorem 11. *Let*

$$1 \longrightarrow \pi_1(S) \longrightarrow \Gamma \xrightarrow{\ell} G \longrightarrow 1 \tag{1.2}$$

be a short exact sequence with Γ a hyperbolic group, and equip Γ and G with word metrics on finite generating sets. There is a $W > 0$ such that, given any nonfilling γ in $\pi_1(S)$ and any γ -quasiinvariant geodesic \mathcal{G} in Γ , we have $\text{diam}(\ell(\mathcal{G})) \leq W$.

Given an infinite cyclic subgroup of G , one obtains a short exact sequence

$$1 \longrightarrow \pi_1(S) \longrightarrow \Gamma_{\mathbb{Z}} \longrightarrow \mathbb{Z} \longrightarrow 1$$

that injects into (1.2), and one may be tempted to argue that Theorem 11 thus follows quickly from Theorem 2. This attack is thwarted by the fact that $\Gamma_{\mathbb{Z}}$ is wildly metrically distorted in Γ .

Again, the authors and S. Dowdall apply Theorem 11 to prove the following theorem.

Theorem 4 (Dowdall–Kent–Leininger [7]). *Let*

$$1 \longrightarrow \pi_1(S) \longrightarrow \Gamma \longrightarrow G \longrightarrow 1$$

be a short exact sequence with Γ hyperbolic. Any quasiconvex finitely generated purely pseudo-Anosov subgroup of $\Gamma \subset \text{Mod}(\mathring{S})$ is convex cocompact. \square

Acknowledgments. The authors thank Ian Agol, Jeff Brock, Dick Canary and Yair Minsky for helpful conversations.

2 Criterion to fill

If M is manifold, $\ell: M \rightarrow \mathbb{R}$ is a function, and X is a subset of M , we define the *width of X with respect to ℓ* (or simply the *width of X*) to be $\text{diam}(\ell(X))$. If X is a subset of any covering space of $\Pi: N \rightarrow M$, we define the *width of X* to be $\text{diam}(\ell(\Pi(X)))$.

Let S be a closed orientable hyperbolic surface. A closed curve in $S \times \mathbb{R}$ is *filling* if its projection to S is filling.

If $M = S \times \mathbb{R}$ is equipped with a hyperbolic metric and Y is an incompressible subsurface of S , we let Γ_Y be the Kleinian group corresponding to $\pi_1(Y) \subset \pi_1(M)$, and $\Pi: M_Y = \mathbb{H}^3/\Gamma_Y \rightarrow M$ the corresponding cover. We let $\mathcal{C}_Y \subset M_Y$ denote the convex core.

Theorem 5. *Let S be a closed oriented surface of euler characteristic $\chi = \chi(S) < 0$ and let ε and K be positive numbers. There is a $W = W(\chi, \varepsilon, K) > 0$ such that the following holds. Equip $M = S \times \mathbb{R}$ with any ε -thick hyperbolic structure, and let $\ell: M \rightarrow \mathbb{R}$ be a K -Lipschitz submersion. If Y is a proper incompressible subsurface of S , then the width of \mathcal{C}_Y is at most W . In particular, if γ is a geodesic loop in M such that $\text{diam}(\ell(\gamma)) > W$, then γ fills S .*

When M is the cover of a fibered hyperbolic 3-manifold corresponding to the fiber, the following lemma follows from the main theorem of [22].

Lemma 6. *If $M = S \times \mathbb{R}$ is equipped with a hyperbolic structure without parabolics, and Y is a proper incompressible subsurface of S , then the group Γ_Y is a Schottky group.*

Proof of Lemma 6. Suppose that Γ_Y is not Schottky a group.

Since M has no cusps, and Π is a covering, M_Y also has no cusps. So Γ_Y must be geometrically infinite.

If we let S_Y denote the covering of S corresponding to Y (which is homeomorphic to the interior of Y), then $M_Y \cong S_Y \times \mathbb{R}$ is homeomorphic to the interior of a handlebody. By Canary's Covering Theorem [6], there is a neighborhood \mathcal{E} of the end of M_Y such that $\Pi|_{\mathcal{E}}$ is finite-to-one. Since Π is a covering map and $M_Y - \mathcal{E}$ is compact, we conclude that Π is finite-to-one. But M is homotopy equivalent to a closed surface and Γ_Y is free. \square

Proof of Theorem 5. Let ∂Y^* be the geodesic representative of ∂Y in M .

The geodesic ∂Y^* is realized by a pleated surface $\mathcal{F} \rightarrow M$ (see Theorem 5.3.6 of [5]). Since M is ε -thick and $\mathcal{F} \rightarrow M$ is a 1-Lipschitz incompressible map, there is a number $B = B(\chi, \varepsilon)$ that bounds the diameter of (the image of) \mathcal{F} in M . Since ℓ is K -Lipschitz the width of \mathcal{F} is at most KB , and hence so is ∂Y^* .

If \mathcal{C}_Y has no interior, we let $\partial \mathcal{C}_Y$ be the double $\mathfrak{D}\mathcal{C}_Y$, considered as a map $\mathfrak{D}\mathcal{C}_Y \rightarrow \mathcal{C}_Y \rightarrow M$. Note that since Γ_Y is Schottky, $\partial \mathcal{C}_Y$ is a nonempty, compact pleated surface.

Lemma 7. *There is a number $W = W(\chi, \varepsilon, K)$ such that $\partial \mathcal{C}_Y$ has width less than W .*

Proof. Let δ be less than the minimum of ε and the 2-dimensional Margulis constant.

There is a number $D = D(\chi, \varepsilon)$ such that $\partial \mathcal{C}_Y$ lies in the D -neighborhood of ∂Y^* . To see this, let $\mathcal{P}(\delta)$ be the δ -thin part of $\partial \mathcal{C}_Y$, and note that the components of $\partial \mathcal{C}_Y - \mathcal{P}(\delta)$ have diameters bounded above by a constant $E = E(\chi, \delta)$. Since M is ε -thick, every loop in $\mathcal{P}(\delta)$ bounds a disk in M_Y . Moreover, every point in $\mathcal{P}(\delta)$ lies in a loop of length less than δ . Such a loop bounds a disk in M_Y of diameter at most δ , and since ∂Y^* is disk-busting, every point of $\mathcal{P}(\delta)$ is within δ of ∂Y^* . But every point of $\partial \mathcal{C}_Y - \mathcal{P}(\delta)$ is within E of $\mathcal{P}(\delta)$. Letting $D = E + \delta$, we have $\partial \mathcal{C}_Y$ contained in the D -neighborhood of ∂Y^* .

Since ∂Y^* has width at most KB , the width of $\partial \mathcal{C}_Y$ is at most $W = KB + 2KD$. \square

If $\partial \mathcal{C}_Y = \mathcal{C}_Y$, we are done by Lemma 7. So we assume that $\mathcal{C}_Y^\circ \neq \emptyset$. The map $\pi: \mathcal{C}_Y \rightarrow M$ is an immersion on \mathcal{C}_Y° , and since ℓ is a submersion, the composition $\ell \circ \pi: \mathcal{C}_Y \rightarrow \mathbb{R}$ is a submersion on \mathcal{C}_Y° as well. It follows that $\ell \circ \pi$ achieves its extrema on $\partial \mathcal{C}_Y$. So the width of \mathcal{C}_Y equals the width of $\partial \mathcal{C}_Y$, which is bounded by Lemma 7. \square

3 The cusped case

Let S be a noncompact finite-volume hyperbolic surface with euler characteristic $\chi < 0$, and let M be a hyperbolic manifold homeomorphic to $S \times \mathbb{R}$. Note that simply projecting to \mathbb{R} is not Lipschitz, and, as such projections are natural for measuring width, we find the naive analog of Theorem 5 too restrictive. In this section, we discuss the correct analog.

Let $M = S \times \mathbb{R}$, and equip M with a type-preserving hyperbolic structure without accidental parabolics. Let $P \subset S$ denote standard cusp neighborhoods of the ends, so that $S^0 = S - P$ is a compact surface with boundary and $S^0 \rightarrow S$ is a homotopy equivalence. Let $\mathbf{P} = P \times \mathbb{R} \subset M$ and set

$$M^0 = M - \mathbf{P} = S^0 \times \mathbb{R}.$$

We assume that the restriction of the hyperbolic metric to each component of \mathbf{P} is isometric to a standard cusp neighborhood

$$\mathbf{P}_3(r) = \{(z, t) \in \mathbb{H}^3 \mid t > r\} / \langle (z, t) \mapsto (z + 1, t) \rangle,$$

for some r satisfying $\operatorname{arccosh}(1 + 1/2r^2) < \mu_3$, where μ_3 is the 3-dimensional Margulis constant. We often write $\mathbf{P}(r) = \mathbf{P}$ when r is relevant.

Given an essential subsurface $Y \subset S$, let $M_Y \rightarrow M$ denote the cover corresponding to Y and $\mathcal{C}_Y \subset M_Y$ its convex core. An argument similar to the proof of Lemma 6 shows that the Kleinian group Γ_Y corresponding to Y is geometrically finite without accidental parabolics. The boundary $\partial\mathcal{C}_Y$ is a locally convex pleated surface whose cusps are carried to cusps of M_Y (consequently, \mathcal{C}_Y is bent along a compact geodesic lamination). Each cusp of $\partial\mathcal{C}_Y$ has a *standard neighborhood* \mathcal{U}_r isometric to

$$\mathbf{P}_2(r) = \{(x, t) \in \mathbb{H}^2 \mid t > r\} / \langle (x, t) \mapsto (x + 1, t) \rangle.$$

There is an $r_0 = r_0(\chi)$ such that \mathcal{U}_r is disjoint from the pleating locus when $r \geq r_0$ (as there is a definite cusp neighborhood in any hyperbolic surface that misses every compact geodesic lamination). It follows that, for $r \geq r_0$, our \mathcal{U}_r is totally geodesic. We take $r \geq \max\{r_0, (2 \cosh(\mu_3) - 1)^{-1/2}\}$, thus ensuring that \mathcal{U}_r is totally geodesic and carried into \mathbf{P} .

Proposition 8. *There is an $r = r(\chi)$ with the following property. Equip $M = S \times \mathbb{R}$ with a type-preserving hyperbolic metric without accidental parabolics, and suppose each component of \mathbf{P} is isometric to $\mathbf{P}_3(r)$. Let $Y \subset S$ be an essential subsurface whose corresponding cover $M_Y \rightarrow M$ has convex core \mathcal{C}_Y . Then each component of the intersection of \mathcal{C}_Y and \mathbf{P} is isometric to*

$$\mathbf{P}_3(r, R) = \{(z, t) \in \mathbb{H}^3 \mid t > r \text{ and } 0 \leq \operatorname{Im}(z) \leq R\} / \langle (z, t) \mapsto (z + 1, t) \rangle$$

for some $R > 0$.

Proof. An area argument shows that if $r > 0$ is sufficiently large (depending only on χ), any pleated surface representative of S meets $\mathbf{P}(r)$ only in its cusps, and we assume that r is at least this large, in addition to the constraints already imposed on r .

Let Y be an essential subsurface of S . For a given $r > 0$, let \mathcal{V}_r be the union of the cusp neighborhoods $\mathcal{U}_r \subset \partial\mathcal{C}_Y$ constructed above. If $r > 0$ is sufficiently large, and a point of $\partial\mathcal{C}_Y - \mathcal{V}_r$ is sufficiently deep in $\mathbf{P}(r)$, then area considerations again imply that $\partial\mathcal{C}_Y - \mathcal{V}_r$ must contain a compressible curve bounding a disk \mathcal{D} contained in \mathcal{C}_Y and some component of $\mathbf{P}(r)$. Since ∂Y is disk-busting in \mathcal{C}_Y , its geodesic representative $\partial Y^* \subset \mathcal{C}_Y$ must intersect \mathcal{D} , and hence $\mathbf{P}(r)$. But this means that if $\mathcal{F} \rightarrow M$ is any pleated surface representative of S realizing ∂Y^* , then the noncuspidal part of \mathcal{F} must hit $\mathbf{P}(r)$, contradicting our choice of r . We find that $\partial\mathcal{C}_Y - \mathcal{V}_r$ is carried a uniformly bounded distance (depending only on χ) into $\mathbf{P}(r)$. Choosing a larger r , we assume that $\partial\mathcal{C}_Y$ hits $\mathbf{P}(r)$ only in the \mathcal{U}_r .

Let $\mathbf{P}_Y(r)$ be the preimage of $\mathbf{P}(r)$ in M_Y . Suppose \mathcal{K} is a component of $\mathcal{C}_Y \cap \mathbf{P}_Y(r)$ which is not of the form $\mathbf{P}_3(r, R)$ for any $R > 0$. Then the closure of \mathcal{K} must intersect $\partial\mathbf{P}_Y(r)$ in a locally convex (horospherical) surface \mathcal{H} . This surface lies in \mathcal{C}_Y° , since

$\partial\mathcal{C}_Y$ hits $\mathbf{P}_Y(r)$ only in the \mathcal{U}_r . Moreover, \mathcal{H} is compact, as \mathcal{C}_Y is compact after its cuspidal thin-part is thrown away. But this all implies that $\partial\mathbf{P}_Y(r)$ in M_Y has a compact component, namely \mathcal{H} , which is absurd. We conclude that every component of $\mathcal{C}_Y \cap \mathbf{P}_Y(r)$ has the form $\mathbf{P}_3(r, R)$. It follows that every component of $\mathcal{C}_Y \cap \mathbf{P}(r)$ has this form. \square

We say that a hyperbolic structure on a noncompact manifold M is ε -**thick** if the length of its shortest geodesic loop is at least ε .

Theorem 9. *Let S be a finite-type noncompact oriented surface of euler characteristic $\chi < 0$. Let ε and K be positive numbers. Equip $M = S \times \mathbb{R}$ with an ε -thick hyperbolic metric, and let $r = r(\chi)$ be the number given by Proposition 8. There is a $W = W(\chi, \varepsilon, K) > 0$ such that the following holds. Let $\ell: M - \mathbf{P}(r) \rightarrow \mathbb{R}$ be a K -Lipschitz map and let $v: M \rightarrow M - \mathbf{P}(r)$ be the normal projection. If Y is a proper incompressible subsurface of S with convex core \mathcal{C}_Y mapping to M via $\Pi: \mathcal{C}_Y \rightarrow M$, then $\text{diam}(\ell(v(\Pi(\mathcal{C}_Y)))) \leq W$. If γ is a geodesic loop in M with $\text{diam}(\ell(v(\gamma))) > W$, then γ fills S .*

We define the *width* of a subset $X \subset M$ to be $\text{diam}(\ell(v(X)))$, and of a subset $X \subset N$ of a covering space $\Pi: N \rightarrow M$ to be $\text{diam}(\ell(v(\Pi(X))))$.

Lemma 10. *There is a constant $W = W(\chi, \varepsilon, K)$ such that $\partial\mathcal{C}_Y$ has width less than W . In particular, the boundary $\partial\mathcal{X}_Y$ of $\mathcal{X}_Y = \mathcal{C}_Y - \mathbf{P}(r)$ has width less than W .*

Proof. The proof is similar to the proof of Lemma 7.

Let δ be the minimum of ε and $\text{arccosh}(1 + 1/2r^2) < \mu_3$.

We again let $\partial\mathcal{C}_Y$ be the double $\mathfrak{D}\mathcal{C}_Y$ when \mathcal{C}_Y is 2-dimensional, considered as a map $\mathfrak{D}\mathcal{C}_Y \rightarrow \mathcal{C}_Y \rightarrow M$.

There is a $D = D(\chi, \varepsilon)$ such that $\partial\mathcal{C}_Y - \mathbf{P}(r)$ lies in the D -neighborhood of ∂Y^* . To see this, let $\mathcal{P}(\delta)$ be the δ -thin part of $\partial\mathcal{C}_Y$. Note that, by our choice of δ , we have $\partial\mathcal{C}_Y - \mathbf{P}(r) \subset \partial\mathcal{C}_Y - \mathcal{P}(\delta)$.

The components of $\partial\mathcal{C}_Y - \mathcal{P}(\delta)$ have diameters uniformly bounded above by a constant $E = E(\chi, \delta)$.

The thin part $\mathcal{P}(\delta)$ is a union of cusp-neighborhoods and neighborhoods of short geodesics. The cusp neighborhoods lie in $\mathbf{P}(r)$. As before, the geodesic neighborhoods are within δ of the disk-busting ∂Y^* .

We conclude that $\partial\mathcal{C}_Y - \mathbf{P}(r)$ is contained in the D -neighborhood of ∂Y^* for $D = E + \delta$.

Since ∂Y^* has width at most KB , the width of $\partial\mathcal{C}_Y$, which is equal to the width of $\partial\mathcal{C}_Y - \mathbf{P}(r)$, is at most $W = KB + 2KD$. \square

Proof of Theorem 9. The proof is essentially the same as Theorem 5. If \mathcal{C}_Y° is empty, then $\mathcal{C}_Y = \partial\mathcal{C}_Y$ and the theorem follows immediately from Lemma 10. When $\mathcal{C}_Y^\circ \neq \emptyset$, we first observe that by the definition of v and Proposition 8

$$\text{diam}(\ell(v(\Pi(\mathcal{C}_Y)))) = \text{diam}(\ell(\Pi(\mathcal{C}_Y - \mathbf{P}(r)))).$$

The composition $\ell \circ \Pi$ restricted to $\mathcal{C}_Y^\circ - \mathbf{P}(r)$ is a submersion and hence on $\mathcal{C}_Y - \mathbf{P}(r)$ attains its maximum and minimum values on $\partial\mathcal{X}_Y$. By Lemma 10, the width of \mathcal{C}_Y is at most W . \square

4 General surface bundles

We again assume that S is a *closed* surface.

Consider a short exact sequence $1 \rightarrow \pi_1(S) \rightarrow \Gamma \rightarrow G \rightarrow 1$ where Γ is hyperbolic, which we call a *hyperbolic sequence*. We choose a finite generating set for Γ containing one for $\pi_1(S)$, which in turn provides one for G , and we let $X_{\pi_1(S)}$, X_Γ , X_G be the corresponding Cayley graphs. As X_G is of primary importance, we often write $X = X_G$. There are simplicial maps

$$X_{\pi_1(S)} \longrightarrow X_\Gamma \xrightarrow{\pi} X_G$$

which induce our short exact sequence. For any γ in Γ , we let $\tilde{\gamma}^*$ denote any geodesic in X_Γ whose endpoints are the ideal fixed points of γ . So $\tilde{\gamma}^*$ is a γ -quasiinvariant geodesic.

Theorem 11. *Given a hyperbolic sequence $1 \rightarrow \pi_1(S) \rightarrow \Gamma \rightarrow G \rightarrow 1$, there is a $W > 0$ such that, given any nonfilling γ in $\pi_1(S)$ and any γ -quasiinvariant geodesic $\tilde{\gamma}^*$, we have $\text{diam}(\pi(\tilde{\gamma}^*)) \leq W$.*

The statement needed in [7] is the following, which follows easily from Theorem 11. Given a hyperbolic sequence $1 \rightarrow \pi_1(S) \rightarrow \Gamma \rightarrow G \rightarrow 1$ and a proper subsurface $Y \subset S$ with associated subgroup $\Gamma_Y < \Gamma$, we let $\text{WH}(\Gamma_Y)$ denote the union of all quasiinvariant geodesic axes of elements in Γ_Y .

Corollary 12. *Given a hyperbolic sequence $1 \rightarrow \pi_1(S) \rightarrow \Gamma \rightarrow G \rightarrow 1$, there is a $W' > 0$ such that, given any proper subsurface $Y \subset S$ with corresponding subgroup $\Gamma_Y < \Gamma$ we have $\text{diam}(\pi(\text{WH}(\Gamma_Y))) \leq W'$.*

Proof. Let W be as in Theorem 11, let δ be the hyperbolicity constant for Γ , and set $W' = W + 4\delta$. Given two elements γ_1 and γ_2 in Γ , let $\tilde{\gamma}_1^*$ and $\tilde{\gamma}_2^*$ be a pair of respective quasiinvariant geodesics. It suffices to show that $\text{diam}(\pi(\tilde{\gamma}_1^* \cup \tilde{\gamma}_2^*)) \leq W'$, since the diameter of $\pi(\text{WH}(\Gamma_Y))$ is bounded by the supremum of such diameters over all pairs of quasiinvariant axes for all pairs of elements in Γ_Y .

We choose points $x_i \in \tilde{\gamma}_i^*$ with $\text{diam}(\pi(x_1 \cup x_2)) = \text{diam}(\pi(\tilde{\gamma}_1^* \cup \tilde{\gamma}_2^*))$. Applying γ_i to $\tilde{\gamma}_i^*$ for $i = 1, 2$, we assume that x_1 and x_2 are far from $\tilde{\gamma}_2^*$ and $\tilde{\gamma}_1^*$, respectively. There is then a third element γ_3 in Γ_Y with a quasiinvariant geodesic $\tilde{\gamma}_3^*$ that contains x_1 and x_2 in its 2δ -neighborhood $\mathcal{N}_{2\delta}(\tilde{\gamma}_3^*)$. Since γ_3 is in Γ_Y and π is 1-Lipschitz, Theorem 11 gives us

$$\text{diam}(\pi(\tilde{\gamma}_1^* \cup \tilde{\gamma}_2^*)) = \text{diam}(\pi(x_1 \cup x_2)) \leq \text{diam}(\mathcal{N}_{2\delta}(\tilde{\gamma}_3^*)) \leq W + 4\delta = W'. \quad \square$$

Let $\rho: G \rightarrow \text{Mod}(S)$ be the monodromy representation. By [8], hyperbolicity of the sequence implies that ρ has finite kernel and that $G_0 = \rho(G)$ is a *convex cocompact* subgroup of $\text{Mod}(S)$, meaning that G_0 has a quasiconvex orbit in Teichmüller space.

The preimage of G_0 in $\text{Mod}(S)$ is an extension Γ_{G_0} of G_0 by $\pi_1(S)$, which is the

homomorphic image of Γ , and we have the commutative diagram with exact rows

$$\begin{array}{ccccccc}
1 & \longrightarrow & \pi_1(S) & \longrightarrow & \text{Mod}(\mathring{S}) & \longrightarrow & \text{Mod}(S) \longrightarrow 1 \\
& & \parallel & & \uparrow & & \uparrow \\
1 & \longrightarrow & \pi_1(S) & \longrightarrow & \Gamma_{G_0} & \longrightarrow & G_0 \longrightarrow 1 \\
& & \parallel & & \uparrow & & \uparrow \\
1 & \longrightarrow & \pi_1(S) & \longrightarrow & \Gamma & \longrightarrow & G \longrightarrow 1
\end{array}$$

The map $\Gamma \rightarrow \Gamma_{G_0}$ also has finite kernel, and is thus a quasiisometry. Using stability of geodesics in Gromov hyperbolic spaces, one can easily check that it suffices to prove Theorem 11 when $\rho: G \rightarrow G_0$ is an isomorphism. We therefore assume that G is a convex cocompact subgroup of $\text{Mod}(S)$ and that $\Gamma = \Gamma_G = \Gamma_{G_0}$.

There is a canonical S -bundle $\mathcal{S}(S)$ over Teichmüller space $\mathcal{T}(S)$ in which the fiber over $[m]$ in $\mathcal{T}(S)$ is identified with S endowed with the hyperbolic metric m . The universal cover of this space is a hyperbolic plane bundle $\mathcal{H}(S) \rightarrow \mathcal{T}(S)$. The Bers fibration [3] identifies $\mathcal{H}(S)$ and the Teichmüller space $\mathcal{T}(\mathring{S})$ of \mathring{S} , and we have the commutative diagram with equivariant actions

$$\begin{array}{ccccc}
1 & \longrightarrow & \pi_1(S) & \longrightarrow & \text{Mod}(\mathring{S}) \longrightarrow \text{Mod}(S) \longrightarrow 1 \\
& & \circ & & \circ & & \circ \\
\mathbb{H}^2 & \longrightarrow & \mathcal{H}(S) & \longrightarrow & \mathcal{T}(S) \\
\downarrow & & \downarrow & & \downarrow \\
S & \longrightarrow & \mathcal{S}(S) & \longrightarrow & \mathcal{T}(S)
\end{array}$$

We fix a connection on $\mathcal{S}(S) \rightarrow \mathcal{T}(S)$, meaning that we choose smoothly varying direct-sum decomposition of each tangent space of $\mathcal{S}(S)$ into the tangent space of the fiber and a choice of *horizontal space*.

We pick a G -equivariant embedding $X = X_G \rightarrow \mathcal{T}(S)$ which sends edges to geodesics, and which is therefore Lipschitz. We have pullback bundles

$$\begin{array}{ccccc}
\mathbb{H}^2 & \longrightarrow & \mathcal{H}_X & \longrightarrow & X \\
\downarrow & & \downarrow & & \parallel \\
S & \longrightarrow & \mathcal{S}_X & \longrightarrow & X
\end{array}$$

and we call $\mathbb{H}^2 \rightarrow \mathcal{H}_X \rightarrow X$ an *associated hyperbolic plane bundle*. For x in X , we let \mathcal{H}_x denote the fiber of $\mathcal{H}_X \rightarrow X$ over x . We let π stand for any of the maps $\mathcal{H}_X \rightarrow X$, $\mathcal{S}_X \rightarrow X$, and $X_\Gamma \rightarrow X$, letting context determine which is meant.

Pulling our connection back to \mathcal{S}_X , we equip \mathcal{S}_X with a piecewise Riemannian metric that locally splits as a product of the hyperbolic metric on the fibers and the metric lifted from X . We pull this metric back to \mathcal{H}_X .

Given two points x and y in X and a geodesic between them, there is a parallel transport map $\mathcal{H}_x \rightarrow \mathcal{H}_y$ defined by following the horizontal lines of the connection

over the geodesic. Since G acts cocompactly on X , there is a $K_0 > 0$ so that for any two points x and y in X , this map is $K_0^{d(x,y)}$ -bilipschitz with respect to the hyperbolic metrics on the fibers.

There is a fiber-preserving Γ -equivariant quasiisometry $X_\Gamma \rightarrow \mathcal{H}_X$ making the following diagram commute:

$$\begin{array}{ccccc} X_{\pi_1(S)} & \longrightarrow & X_\Gamma & \longrightarrow & X \\ \downarrow & & \downarrow & & \parallel \\ \mathbb{H}^2 & \longrightarrow & \mathcal{H}_X & \longrightarrow & X \end{array}$$

Given γ in $\pi_1(S)$, let $\mathcal{A}_x(\gamma)$ denote the axis of γ in the fiber \mathcal{H}_x and define a subset $\mathcal{A}(\gamma)$ of \mathcal{H}_X by

$$\mathcal{A}(\gamma) = \bigcup_{x \in X} \mathcal{A}_x(\gamma).$$

Let x_γ in X be a point for which the translation length of γ on $\mathcal{A}_{x_\gamma}(\gamma)$ is minimal over all $\mathcal{A}_x(\gamma)$. We endow $\mathcal{A}(\gamma)$ with the subspace metric coming from the path metric on the 1-neighborhood $\mathcal{N}_1(\mathcal{A}(\gamma))$, and denote both of these metrics by d_γ .

By the stability of geodesics in hyperbolic spaces, the following theorem implies Theorem 11.

Theorem 13. *Given a hyperbolic sequence $1 \rightarrow \pi_1(S) \rightarrow \Gamma \rightarrow G \rightarrow 1$ with associated hyperbolic plane bundle $\mathbb{H}^2 \rightarrow \mathcal{H}_X \rightarrow X$, there exist $K, C > 0$ such that if γ in $\pi_1(S)$ is a nonfilling loop in S , then $\mathcal{A}_{x_\gamma}(\gamma)$ is a (K, C) -quasigeodesic in \mathcal{H}_X .*

The rest of the paper is devoted to the proof of Theorem 13. We pause to sketch the proof which is inspired by the ideas in [8], [9], [19], [15].

By a result of Mitra (Lemma 14 below), there are $K_1, C_1 > 0$ independent of γ so that $\mathcal{A}(\gamma)$ is (K_1, C_1) -quasiisometrically embedded in \mathcal{H}_X . The space $\mathcal{A}(\gamma)$ is foliated by the axes $\mathcal{A}_x(\gamma)$. Arguments from Farb–Mosher [8] imply that as one moves x away from x_γ , lengths in $\mathcal{A}_x(\gamma)$ grow exponentially compared to the “corresponding” lengths in $\mathcal{A}_{x_\gamma}(\gamma)$. In other words, the leaves of the foliation “flare” as you move away from x_γ . This exponential flaring kicks in outside of an R -neighborhood $B(x_\gamma, R)$ of x_γ for some R depending on γ . It follows that the subset $\mathcal{A}_{B(x_\gamma, R)}(\gamma)$ of $\mathcal{A}(\gamma)$ over $B(x_\gamma, R)$ is quasiconvex in $\mathcal{A}(\gamma)$, and hence in \mathcal{H}_X . However, the quasiconvexity constant depends on R . This is remedied by an argument of Masur–Minsky [15], which provides a single R that is suited to all nonfilling curves.

4.1 Fiberwise projection

The following construction is due to Mitra [18] and is used throughout his work. Consider the map $p_\gamma: \mathcal{H}_X \rightarrow \mathcal{A}(\gamma)$ obtained by fiberwise closest point projection to $\mathcal{A}(\gamma)$. That is, for z in \mathcal{H}_x , let $p_\gamma(z)$ be the point on $\mathcal{A}_x(\gamma)$ which is closest to z with respect to the hyperbolic metric on \mathcal{H}_x . The following Lemma is a translation to our setting of the results in Section 3 of Mitra’s paper [18]. We give the proof for the reader’s convenience.

Lemma 14 (Mitra [18]). *Given a hyperbolic sequence $1 \rightarrow \pi_1(S) \rightarrow \Gamma \rightarrow G \rightarrow 1$ with associated hyperbolic plane bundle $\mathbb{H}^2 \rightarrow \mathcal{H}_X \rightarrow X$, there are $K_1, C_1 > 0$ such that for any γ in $\pi_1(S)$, the projection $\mathbf{p}_\gamma : \mathcal{H}_X \rightarrow \mathcal{A}(\gamma)$ is (K_1, C_1) -coarsely Lipschitz. Consequently, $\mathcal{A}(\gamma)$ is (K_1, C_1) -quasiisometrically embedded in \mathcal{H}_X . \square*

Proof. We begin with a few observations about the metric d_γ . For any $0 < r < 1$ and $x \in X$, consider the r -neighborhood of the fiber over $x \in X$, $\mathcal{N}_r(\mathcal{H}_x) = \pi^{-1}(B(x, r))$. Because $r < 1$, $B(x, r)$ is a tree in X , and so there is a unique parallel transport to the fiber \mathcal{H}_x for every point in $\mathcal{N}_r(\mathcal{H}_x)$. We denote this map

$$\mathbf{f}_x : \mathcal{N}_r(\mathcal{H}_x) \rightarrow \mathcal{H}_x$$

and observe that it is K_0^r -Lipschitz and K_0^r -biLipschitz when restricted to any fiber \mathcal{H}_y , for $y \in B(x, r)$.

Choose $0 < r < 1$ so that the stability constant for $(K_0^r, 0)$ -quasigeodesics in \mathbb{H}^2 is less than 1. For any $x, y \in X$ with $d(x, y) \leq r$, it follows that the parallel transport line from $z \in \mathcal{A}_y(\gamma)$ to $\mathbf{f}_x(z) \in \mathcal{H}_x$ is contained in $\mathcal{N}_1(\mathcal{A}(\gamma))$ and hence

$$d_\gamma(z, \mathbf{f}_x(z)) = d(z, \mathbf{f}_x(z)) = d(x, y) \leq r.$$

Let δ_h denote the hyperbolicity constant for \mathbb{H}^2 .

Claim 15. *Given any two points $w, z \in \mathcal{H}_X$ with $d(w, z) \leq r$, we have*

$$d_\gamma(\mathbf{p}_\gamma(w), \mathbf{p}_\gamma(z)) \leq K_0^r r + 2(1 + K_0^r \delta_h) + r.$$

Proof of claim. Let $w, z \in X$ be any two points with $d(w, z) \leq r$ and let $x = \pi(w)$ and $y = \pi(z)$ so that $d(x, y) \leq r$.

Recall that for any $c \geq \delta_h$ and any geodesic triangle $\triangle \subset \mathbb{H}^2$, the set of points within a distance c of all three sides is nonempty and has diameter at most $2c$. The closest point projection of one vertex of \triangle to the opposite side is such a point.

Inside \mathcal{H}_y , the point $\mathbf{p}_\gamma(z)$ is within δ_h of all three sides of the geodesic triangle \triangle having vertex z and opposite side $\mathcal{A}_y(\gamma)$. It follows that inside \mathcal{H}_x , the point $\mathbf{f}_x \mathbf{p}_\gamma(z)$ has distance at most $K_0^r \delta_h$ from all three sides of the $(K_0^r, 0)$ -quasigeodesic triangle $\mathbf{f}_x(\triangle)$. Because the sides of this are within a distance 1 of the geodesics with the same endpoints, it follows that $\mathbf{f}_x \mathbf{p}_\gamma(z)$ is within a distance $1 + K_0^r \delta_h$ of all three sides of the geodesic triangle defined by $\mathbf{f}_x(z)$ and $\mathcal{A}_x(\gamma)$. Since $\mathbf{p}_\gamma \mathbf{f}_x(z)$ has distance at most $\delta_h < 1 + K_0^r \delta_h$ from each of these sides, it follows that

$$d_x(\mathbf{p}_\gamma \mathbf{f}_x(z), \mathbf{f}_x \mathbf{p}_\gamma(z)) \leq 2(1 + K_0^r \delta_h).$$

Moreover, the path exhibiting this distance bound lies entirely inside \mathcal{H}_x , and the geodesic in \mathcal{H}_x between these points lies within a distance 1 of $\mathcal{A}_x(\gamma)$. In particular, it follows that

$$d_\gamma(\mathbf{p}_\gamma \mathbf{f}_x(z), \mathbf{f}_x \mathbf{p}_\gamma(z)) \leq d_x(\mathbf{p}_\gamma \mathbf{f}_x(z), \mathbf{f}_x \mathbf{p}_\gamma(z)) \leq 2(1 + K_0^r \delta_h).$$

Applying the triangle inequality proves the claim, since

$$\begin{aligned} d_\gamma(\mathbf{p}_\gamma(w), \mathbf{p}_\gamma(z)) &\leq d_\gamma(\mathbf{p}_\gamma(w), \mathbf{p}_\gamma f_x(z)) + d_\gamma(\mathbf{p}_\gamma f_x(z), f_x \mathbf{p}_\gamma(z)) \\ &\quad + d_\gamma(f_x \mathbf{p}_\gamma(z), \mathbf{p}_\gamma(z)) \end{aligned} \quad (4.1)$$

$$\leq d_x(w, f_x(z)) + 2(1 + K_0^r \delta_h) + r \quad (4.2)$$

$$\leq K_0^r d(w, z) + 2(1 + K_0^r \delta_h) + r \quad (4.3)$$

$$\leq K_0^r r + 2(1 + K_0^r \delta_h) + r. \quad (4.4)$$

In inequality (4.3), we have used the fact that f_x is K_0^r -Lipschitz. \square

From the claim we see that \mathbf{p}_γ is (K_1, C_1) -coarsely Lipschitz, where $K_1 = K_0^r + 2(1 + K_0^r \delta_h)/r + 1$ and $C_1 = K_0^r r + 2(1 + K_0^r \delta_h) + r$. Since the inclusion of $\mathcal{A}(\gamma)$ into \mathcal{H}_X is 1-Lipschitz, it follows that $\mathcal{A}(\gamma)$ is (K_1, C_1) -quasiisometrically embedded. \square

4.2 Quasiisometric sections

Let E and B be metric spaces and let $\pi: E \rightarrow B$ be a 1-Lipschitz map. By a (k, c) -quasiisometric section (or just (k, c) -section) of $\pi: E \rightarrow B$ we mean a subset $\Sigma \subset E$ that is the image of a (k, c) -coarsely Lipschitz map $\sigma: B \rightarrow E$ with $\pi \circ \sigma = id_B$. Since π is 1-Lipschitz, the map σ is a (k, c) -quasiisometric embedding. In fact,

$$d(x, y) = d(\pi\sigma(x), \pi\sigma(y)) \leq d(\sigma(x), \sigma(y)) \leq kd(x, y) + c.$$

Mosher's Quasiisometric Section Lemma [20] says that if $1 \rightarrow \pi_1(S) \rightarrow \Gamma \rightarrow G \rightarrow 1$ is hyperbolic, then there is a (k_0, c_0) -section of $\pi: X_\Gamma \rightarrow X$ for some k_0 and c_0 . From this we obtain a (k_0, c_0) -section Σ of $\mathcal{H}_X \rightarrow X$ after enlarging k_0 and c_0 . Using the fact that $\pi_1(S) < \Gamma$ acts cocompactly on the fibers, and by taking c_0 even larger, it follows that for any point z in \mathcal{H}_X there is a (k_0, c_0) -section Σ for $\mathcal{H}_X \rightarrow X$ containing z ; see also [19].

Given a (k_0, c_0) -section Σ of $\mathcal{H}_X \rightarrow X$, we have that $\mathbf{p}_\gamma(\Sigma)$ is a (K_2, C_2) -section for $K_2 = k_0 K_1$ and $C_2 = K_1 c_0 + C_1$, by Lemma 14. We therefore have the following result of [19].

Lemma 16 (Mj–Sardar [19]). *Given a hyperbolic sequence $1 \rightarrow \pi_1(S) \rightarrow \Gamma \rightarrow G \rightarrow 1$ with associated hyperbolic plane bundle $\mathbb{H}^2 \rightarrow \mathcal{H}_X \rightarrow X$, there are K_2 and C_2 with the following property. For all γ in $\pi_1(S)$, all x in X , and all z in $\mathcal{A}_x(\gamma)$ there exists a (K_2, C_2) -section Σ of $\mathcal{H}_X \rightarrow X$ with $\Sigma \subset \mathcal{A}(\gamma)$ and $\Sigma \cap \mathcal{H}_x = \{z\}$.* \square

A section Σ as in this lemma will be called a (K_2, C_2) -section for γ (though z). In the sequel we are interested in collections of these. The leaf $\mathcal{A}_x(\gamma)$ is a line oriented by the action of γ , and so possesses a well-defined order. We say that a collection $\{\Sigma_n\}_{n \in \mathbb{Z}}$ of (K_2, C_2) -sections for γ are *linearly ordered over x* if the assignment $n \mapsto \Sigma_n \cap \mathcal{A}_x(\gamma)$ is order preserving.

Theorem 17. *Given a hyperbolic sequence $1 \rightarrow \pi_1(S) \rightarrow \Gamma \rightarrow G \rightarrow 1$ with associated hyperbolic plane bundle $\mathbb{H}^2 \rightarrow \mathcal{H}_X \rightarrow X$, there are $D_1 > D_0 > 0$ with the following*

property. If γ in $\pi_1(S)$ is nonfilling and $\{\Sigma_n\}_{n \in \mathbb{Z}}$ is a collection of (K_2, C_2) -sections for γ such that

$$\{\Sigma_n\}_{n \in \mathbb{Z}} \text{ is linearly ordered over } x_\gamma \text{ and } d_{x_\gamma}(\Sigma_n, \Sigma_{n+1}) = D_1,$$

then, for **every** x in X ,

$$\{\Sigma_n\}_{n \in \mathbb{Z}} \text{ is linearly ordered over } x \text{ and } d_x(\Sigma_n, \Sigma_{n+1}) \geq D_0.$$

Proof of Theorem 13 assuming Theorem 17. Let γ be nonfilling. By Lemma 16, there are (K_2, C_2) -sections $\{\Sigma_n\}_{n \in \mathbb{Z}}$ for γ as in Theorem 17.

Let \mathcal{R}_n denote the open region in $\mathcal{A}(\gamma)$ between Σ_n and Σ_{n+1} . By the conclusion of Theorem 17, each \mathcal{R}_n is a union of intervals, one in each fiber. According to Theorem 3.2 of [19], there are constants K' and C' depending only the bundle $\mathbb{H}^2 \rightarrow \mathcal{H}_X \rightarrow X$ such that the fiberwise closest point projection

$$\mathfrak{p}_n: \mathcal{H}_X \rightarrow \mathcal{R}_n$$

is (K', C') -coarsely Lipschitz map (where \mathcal{R}_n is given the metric inherited from the path metric on a sufficiently large neighborhood in \mathcal{H}_X). Theorem 3.2 of [19] is attributed to Mitra [18], as it is a direct translation of arguments there, much like the proof of Lemma 14.

Define

$$\eta_\gamma: \mathcal{A}(\gamma) \rightarrow \mathcal{A}_{x_\gamma}(\gamma)$$

by $\eta_\gamma(\mathcal{R}_n) = \eta_\gamma(\Sigma_n) = \Sigma_n \cap \mathcal{A}_{x_\gamma}(\gamma)$. We will show that η_γ is coarsely Lipschitz.

Claim. *There is a $B_1 > 0$ depending only on the bundle $\mathbb{H}^2 \rightarrow \mathcal{H}_X \rightarrow X$ such that if w is in $\mathcal{R}_m \cup \Sigma_m$ and z is in $\mathcal{R}_n \cup \Sigma_n$ with $d(w, z) \leq 1$, then $|m - n| \leq B_1$.*

Proof of claim. Assume that $m \leq n$.

First assume that w and z are in the same fiber $\mathcal{A}_{\pi(w)}(\gamma) = \mathcal{A}_{\pi(z)}(\gamma)$. By Theorem 17, we have $d_{\pi(w)}(w, z) \geq D_0(n - m)$. Now, the fibers of \mathcal{H}_X (in which the fibers of $\mathcal{A}(\gamma)$ are geodesic) are uniformly proper, and so there is a positive E_0 depending only on $\mathbb{H}^2 \rightarrow \mathcal{H}_X \rightarrow X$ such that $d(w, z) \geq E_0 d_{\pi(w)}(w, z)$. So

$$1 \geq d(w, z) \geq E_0 D_0 (n - m - 1),$$

and we are done in this case with $B_1 = 1/E_0 D_0 + 1$.

If w and z are in different fibers, we argue as follows. Let z' be a point in the fiber $\mathcal{H}_{\pi(w)}$ with

$$d(z, z') = d(z, \mathcal{H}_{\pi(w)}) \leq d(z, w) \leq 1.$$

We have $\mathfrak{p}_n(z) = z$ and $\mathfrak{p}_n(z') = z''$ for some z'' in $\mathcal{R}_n \cap \mathcal{H}_{\pi(w)}$. Since \mathfrak{p}_n is (K', C') -coarsely Lipschitz, uniform properness gives us

$$\begin{aligned} 1 + K' + C' &\geq 1 + K' d(z, z') + C' \\ &\geq d(w, z) + d(z, z'') \\ &\geq d(w, z'') \\ &\geq E_0 D_0 (n - m - 1), \end{aligned}$$

and the proof is complete with $B_1 = (1 + K' + C')/E_0 D_0 + 1$. \square

It follows from the claim that

$$d_{x_\gamma}(\eta_\gamma(z), \eta_\gamma(w)) \leq B_1 D_1$$

if $d(z, w) \leq 1$, and so η_γ is $(B_1 D_1, B_1 D_1)$ -coarsely Lipschitz. It follows that $\mathcal{A}_{x_\gamma}(\gamma)$ is $(B_1 D_1, B_1 D_1)$ -quasiisometrically embedded in $\mathcal{A}(\gamma)$, and hence (K, C) -quasiisometrically embedded in \mathcal{H}_X for $K = K_1 B_1 D_1$ and $C = K_1 B_1 D_1 + C_1$, by Lemma 14. \square

For x sufficiently far from x_γ , the distances $d_x(\Sigma_n, \Sigma_{n+1})$ are in fact much larger than the estimate in Theorem 17. As a function of $d(x, x_\gamma)$, they are exponentially larger than the distances $d_{x_\gamma}(\Sigma_n \cap \mathcal{A}_{x_\gamma}(\gamma), \Sigma_{n+1} \cap \mathcal{A}_{x_\gamma}(\gamma))$, due to *flaring*. For nonfilling γ , the exponential growth will kick in outside a ball about x_γ of a uniformly bounded radius.

The proof of Theorem 17 requires a study of quadratic differentials, Teichmüller geodesics, and singular SOL metrics, which we take up in the next section.

4.3 Quadratic differentials and flat metrics

Given a complex structure on S , a unit-norm holomorphic quadratic differential q on S both determines and is determined by a nonpositively curved Euclidean cone metric on S together with a pair of orthogonal singular foliations with geodesic leaves (called the *vertical* and *horizontal* foliations). Given q and a nonsingular point p , there is a *preferred coordinate* $\zeta = x + iy$ which carries a neighborhood of p isometrically into the plane such that the arcs of the horizontal and vertical foliations to horizontal and vertical segments, respectively.

We let $\mathcal{Q}^1(S)$ denote the space of all unit-norm holomorphic quadratic differentials on S , which forms the unit cotangent bundle over Teichmüller space $\mathcal{T}(S)$. We let $m = m(q)$ denote the hyperbolic metric in the conformal class of a quadratic differential q , and write $q \mapsto m(q)$ for the map $\mathcal{Q}^1(S) \rightarrow \mathcal{T}(S)$.

Let $\tilde{S} \rightarrow S$ be the universal covering. Given q in $\mathcal{Q}^1(S)$, we abuse notation and continue to refer to the pullback of q and m to \tilde{S} as q and m , respectively. The identity map $id_{\tilde{S}}: \tilde{S} \rightarrow \tilde{S}$ is a quasiisometry with respect to m and the singular flat metric for q . In fact, by Proposition 2.5 of [8] or Lemma 3.3 of [16], for example, we have the following lemma.

Lemma 18 (Minsky [16]). *Given $r > 0$ there exist $K_3, C_3 > 0$ such that if q in $\mathcal{Q}^1(S)$ lies over the r -thick part of $\mathcal{T}(S)$, then*

$$id_{\tilde{S}}: (\tilde{S}, m) \rightarrow (\tilde{S}, q)$$

is a (K_3, C_3) -quasiisometry. \square

4.3.1 Geodesics and straight segments.

Fix q in $\mathcal{Q}^1(S)$. Given γ in $\pi_1(S)$ a (nontrivial) element we will let γ_0^* denote the q -geodesic representative in S and $\tilde{\gamma}_0^*$ a lift of this geodesic to a biinfinite q -geodesic in \tilde{S} . The geodesic γ_0^* should be considered a locally isometric map from a circle or interval of some length into S as the geodesic is not determined by its image.

The geodesics γ_0^* and $\tilde{\gamma}_0^*$ are either Euclidean geodesics (geodesics in the complement of the singularities) or concatenations of *straight segments* (Euclidean geodesic segments connecting pairs of singular points with no singular points in their interior).

We let $\|\gamma\|_q$ denote the q -length of γ_0^* and $\|\gamma\|_{q,v}$ and $\|\gamma\|_{q,h}$ the vertical and horizontal lengths of γ_0^* , respectively. These are related by

$$\frac{1}{2}(\|\gamma\|_{q,v} + \|\gamma\|_{q,h}) \leq \max\{\|\gamma\|_{q,v}, \|\gamma\|_{q,h}\} \quad (4.5)$$

$$\leq \|\gamma\|_q \quad (4.6)$$

$$\leq \|\gamma\|_{q,v} + \|\gamma\|_{q,h} \quad (4.7)$$

$$\leq 2 \max\{\|\gamma\|_{q,v}, \|\gamma\|_{q,h}\}. \quad (4.8)$$

More generally, given a (local) q -geodesic $\delta: I \rightarrow S$ or $\delta: I \rightarrow \tilde{S}$ defined on an interval $I \subset \mathbb{R}$, we let $\|\delta\|_q$, $\|\delta\|_{q,h}$, and $\|\delta\|_{q,v}$ denote the length, horizontal length, and vertical length, respectively.

We let $\|\gamma\|_m$ denote the length of the $m = m(q)$ -geodesic representative. Given $r > 0$, if K_3, C_3 are as in Lemma 18, we have

$$\frac{1}{K_3} \|\gamma\|_q \leq \|\gamma\|_m \leq K_3 \|\gamma\|_q. \quad (4.9)$$

The inequality (4.9) is free of the constant C_3 thanks to the fact that the length is equal to the asymptotic translation length.

More generally, given any geodesic metric m' on S for which the pullback to \tilde{S} makes $id_{\tilde{S}}: (\tilde{S}, m') \rightarrow (\tilde{S}, q)$ a (K_6, C_6) -quasiisometry, then

$$\frac{1}{K_6} \|\gamma\|_q \leq \|\gamma\|_{m'} \leq K_6 \|\gamma\|_q. \quad (4.10)$$

From (4.9) we easily obtain the following.

Lemma 19. *For any $r > 0$, there exists $\varepsilon > 0$ with the following property. Given any q in $\mathcal{Q}^1(S)$ lying over the r -thick part of $\mathcal{T}(S)$ and any (local) q -geodesic segment $\delta: [0, 1] \rightarrow S$ or $\delta: [0, 1] \rightarrow \tilde{S}$, there is an arc of δ of length at least ε containing no singularities.*

Proof. We assume as we may that $r < 1$ and set $\varepsilon = r/(K_3(4g-2)) < 1/(4g-2)$.

Suppose that there is a q -geodesic segment $\delta: [0, 1] \rightarrow S$ such that every subsegment of length at least ε contains a singularity. This segment contains a concatenation δ' of at least $4g-4$ straight segments of q -length less than ε , each connecting a pair of singularities. Since there are at most $4g-4$ singularities of q , the segment δ' must visit some singularity more than once, thus forming a loop β of q -length less than $(4g-4)\varepsilon < r/K_3$. Except at the basepoint, this loop β is locally geodesic, and is therefore essential. By (4.9), the hyperbolic length of β is less than $K_3(r/K_3) = r$, which contradicts the fact that q lies over the r -thick part of $\mathcal{T}(S)$.

For $\delta: [0, 1] \rightarrow \tilde{S}$, we push forward to S and appeal to the first case. \square

Applying the lemma to any closed geodesic γ_0 we have the following.

Corollary 20. *Let $r > 0$ and let ε be as in Lemma 19. If q in $\mathcal{Q}^1(S)$ lies over the r -thick part of $\mathcal{T}(S)$ and γ in $\pi_1(S)$, then γ_0 contains a straight segment of length at least ε . \square*

4.4 Teichmüller geodesics and lengths

We refer the reader to [1] and [10] for detailed treatments of Teichmüller theory.

4.4.1 Teichmüller deformations.

The Teichmüller deformation associated to a quadratic differential q in $\mathcal{Q}^1(S)$ determines a 1-parameter family of quadratic differentials q_t . More precisely, if q has preferred coordinate $\zeta = x + iy$, then q_t is determined by its preferred coordinate $\zeta_t = e^t x + i e^{-t} y$ (in particular, $q = q_0$). The map $\tau_q: \mathbb{R} \rightarrow \mathcal{T}(S)$ obtained by composing $t \mapsto q_t$ with the projection $\mathcal{Q}^1(S) \rightarrow \mathcal{T}(S)$, namely $\tau_q(t) = m_t = m(q_t)$, is a *Teichmüller geodesic*. Every geodesic in $\mathcal{T}(S)$ is of this form.

4.4.2 Balance times

If $\delta: I \rightarrow S$ or $\delta: I \rightarrow \tilde{S}$ is a (local) q -geodesic, we can reparameterize δ to be a (local) q_t -geodesic for any t . In particular, straight segments can be linearly reparameterized to be (locally) geodesic. We denote the reparameterization by δ_t .

For any γ in $\pi_1(S)$ we have

$$\|\gamma\|_{q_t, h} = \|\gamma\|_{q, h} e^t \text{ and } \|\gamma\|_{q_t, v} = \|\gamma\|_{q, v} e^{-t}.$$

We let γ_t^* and $\tilde{\gamma}_t^*$ denote the q_t -geodesic reparameterizations of the q_t -geodesics γ_0^* and $\tilde{\gamma}_0^*$, respectively.

We say that γ is *balanced at time t* if $\|\gamma\|_{q_t, h} = \|\gamma\|_{q_t, v}$. If γ is balanced at time t_0 , then for $b = \|\gamma\|_{q_0, v} + \|\gamma\|_{q_0, h}$, we have

$$b \cosh(t - t_0) \leq \|\gamma\|_{q_t} \leq 2b \cosh(t - t_0) \quad (4.11)$$

by (4.5). So $\|\gamma\|_{q_t}$ is minimized in the interval $[t_0 - \operatorname{arccosh}(2), t_0 + \operatorname{arccosh}^{-1}(2)]$ and grows exponentially in $|t|$.

Given any q , suppose m'_t is a 1-parameter family of hyperbolic metrics on S for which $id_{\tilde{S}}: (\tilde{S}, m'_t) \rightarrow (\tilde{S}, q_t)$ is a (K_6, C_6) -quasiisometry. Then

$$\frac{b}{K_6} \cosh(t - t_0) \leq \|\gamma\|_{m'_t} \leq 2bK_6 \cosh(t - t_0) \quad (4.12)$$

by (4.10) and (4.11). In particular, the m'_t -length along $\tau_q(t)$ is minimized in the interval $[t_0 - \operatorname{arccosh}(2K_6^2), t_0 + \operatorname{arccosh}(2K_6^2)]$.

As an example, we could take $m'_t = m_t = m(q_t)$ to be the underlying hyperbolic metric, and then $(K_6, C_6) = (K_3, C_3)$ by Lemma 18. However, Theorem 27 below provides our primary example of interest.

4.4.3 Vertical and horizontal.

Given $\varepsilon > 0$, $0 < \theta < \pi/4$ and q in $\mathcal{Q}^1(S)$, we say that a q -straight segment δ is θ -almost vertical (respectively, θ -almost horizontal) with respect to q if it makes an angle less than θ with the vertical (respectively, horizontal) direction. A closed geodesic γ_0^* , or its lift $\tilde{\gamma}_0^*$, is called (ε, θ) -almost vertical (respectively, (ε, θ) -almost horizontal) with respect to q provided it is a concatenation of q -straight segments each of which is θ -almost vertical (respectively, θ -almost horizontal), or has length less than ε . Subject to certain constraints described below, the constants ε and θ will be fixed, and we will thus refer to segments and geodesics as simply almost vertical or almost horizontal. The discussion here differs from that of [15] in that the constraints we consider depend on the thickness constant $r > 0$.

4.4.4 Nonfilling curves after Masur and Minsky.

The next proposition relies heavily on the work of Masur and Minsky, specifically Sections 5 and 6 of [15]. In particular, we assume henceforth that ε_0, θ_0 are chosen to satisfy Lemmas 6.1–6.5 of [15], as well as the statement of Lemma 19 for $r > 0$. The constants ε_0 and θ_0 depend only on r and χ .

Proposition 21. *Given $r > 0$, there is a $T_r > 0$ with the following property. Suppose q in $\mathcal{Q}^1(S)$ defines an r -thick geodesic τ_q in $\mathcal{T}(S)$ and γ in $\pi_1(S)$ is nonfilling, balanced at time 0 in \mathbb{R} . For any geodesic subpath $\delta_0 \subset \tilde{\gamma}_0^*$ with $\|\delta_0\|_q > e^{T_r}$ we have*

$$\|\delta_t\|_{q_t} > \frac{\varepsilon_0 e^{|t| - T_r}}{4} \|\delta_0\|_q = \frac{\varepsilon_0 e^{-T_r}}{4} e^{|t|} \|\delta_0\|_q$$

for any t .

We note the similarity between the conclusion of this proposition and (4.11). By comparison, (4.11) is a statement about the q_t -length of the entire curve γ , while this proposition provides information about the q_t -length of any definite length segment of γ_0^* . In particular, it also grows exponentially outside some neighborhood of the balance time. Furthermore, while (4.11) is true for any closed geodesic, Proposition 21 is false if one allows γ to be filling: there is no T making the proposition valid for all filling γ .

Proof of Proposition 21. In what follows, we appeal to Lemmas 6.4 and 6.5 of [15], which provide bounds on diameters of shadows in the curve complex $\mathcal{C}(S)$ of certain subsets of the Teichmüller geodesic τ_q . Since ours is an r -thick geodesic, the shadow is a uniform quasigeodesic. This is Lemma 4.4 of [21]. It also follows quickly from the main theorem of [17] (see Section 7.4 of [13]). We may therefore turn bounds on diameters in $\mathcal{C}(S)$ into bounds on diameters in the domain \mathbb{R} of τ_q , and we do so without further comment.

Since γ is nonfilling, there is an essential simple closed curve α disjoint from it. Let t_0 denote the balance time for α .

Claim 22. *There exists $T_0 > 0$, depending only on ε_0, θ_0 , and r such that γ_t^* is almost horizontal for all $t > T_0$ and is almost vertical for all $t < -T_0$.*

Proof. By Lemma 6.5 of [15], there is a $T_1 > t_0$ such that $T_1 - t_0$ is bounded by a constant $B(\varepsilon_0, \theta_0, r)$ and such that for all $t > T_1$, the geodesic α_t^* is almost horizontal. Since $i(\delta, \alpha) = 0$, no segment of γ_t^* intersects any segment of α_t^* away from the singularities. Pick a straight segment of α_t^* with length at least ε_0 (from Corollary 20). As in the last paragraph of the proof of Lemma 6.5 of [15], we can appeal to Lemma 6.4 of [15] to find a $T_2 > T_1$ such that, for all $t > T_2$, the geodesic γ_t^* is almost horizontal.¹ Moreover, the distance $T_2 - T_1$, and hence also $T_2 - t_0$, is bounded by a constant $C(\varepsilon_0, \theta_0, r)$.

Reversing the roles of horizontal and vertical, there is $T_3 < t_0$ such that γ_t^* is almost vertical for all $t < T_3$, and $t_0 - T_3$ is bounded by some $D(\varepsilon_0, \theta_0, r)$. The balance time 0 for γ must occur in the interval $[T_3, T_2]$ (since γ is neither almost vertical nor almost horizontal when it is balanced), and setting $T_0 = \max\{T_2, |T_3|\}$ proves the claim. \square

For all $t > 0$, we have

$$\|\delta_t\|_{q_t} \geq e^{-t} \|\delta_0\|_{q_0}. \quad (4.13)$$

For $t = T_0$, we have

$$\|\delta_{T_0}\|_{q_{T_0}} \geq e^{-T_0} \|\delta_0\|_{q_0},$$

and we set $T_r = 2T_0$.

Now, if $\delta_0 \subset \gamma_{T_0}^*$ is a straight segment of length at least e^{T_r} we have

$$\|\delta_{T_0}\|_{q_{T_0}} \geq e^{-T_0} \|\delta_0\|_{q_0} \geq e^{-T_0} e^{T_r} > 1.$$

Therefore, by Lemma 19, the segment δ_{T_0} contains a segment δ'_{T_0} of length at least ε_0 contained in a straight segment. This segment δ'_{T_0} must be almost horizontal since $\gamma_{T_0}^*$ (and hence $\tilde{\gamma}_{T_0}^*$) is almost horizontal. Therefore, for all $t \geq T_0$ we have

$$\|\delta'_t\|_{q_t} \geq \|\delta'_t\|_{q_t, h} \geq e^{t-T_0} \|\delta'_{T_0}\|_{q_{T_0}, h} \geq \frac{e^{t-T_0}}{2} \|\delta'_{T_0}\|_{q_{T_0}} \geq \frac{\varepsilon_0 e^{t-T_0}}{2}$$

There is such a segment δ'_{T_0} in each segment of length 1 in δ_{T_0} . By subdividing δ_{T_0} into a maximal number n of disjoint segments of length at least 1, so that $n \leq \|\delta_{T_0}\|_{q_{T_0}} < n+1$, we have

$$\|\delta_t\|_{q_t} \geq \frac{n\varepsilon_0 e^{t-T_0}}{2} = \frac{n}{n+1} \frac{(n+1)\varepsilon_0 e^{t-T_0}}{2} \geq \frac{\varepsilon_0 e^{t-T_0}}{4} \|\delta_{T_0}\|_{q_{T_0}}$$

Combining these strings of inequalities we see that, for $t \geq T_0$, we have

$$\|\delta_t\|_{q_t} \geq \frac{\varepsilon_0 e^{t-T_0}}{4} e^{-T_0} \|\delta_0\|_{q_0} = \frac{\varepsilon_0 e^{t-T_r}}{4} \|\delta_0\|_{q_0}.$$

On the other hand, if $0 \leq t < T_0$, then $-t > t - T_r$. Since $\varepsilon_0/4 < 1$, we therefore have

$$\|\delta_t\|_{q_t} \geq e^{-t} \|\delta_0\|_{q_0} \geq e^{t-T_r} \|\delta_0\|_{q_0} \geq \frac{\varepsilon_0 e^{t-T_r}}{4} \|\delta_0\|_{q_0}.$$

by (4.13). Thus the proposition follows for $t \geq 0$. A symmetric argument proves the proposition for $t \leq 0$. \square

¹The key to the proof of Lemma 6.5 of [15] is finding a disjoint almost horizontal straight segment. In our setting, this is provided by a segment of α_t^* .

4.5 Surface bundles over Teichmüller geodesics

4.5.1 Singular SOL and hyperbolic metrics are uniformly quasiisometric

Given q in $\mathcal{Q}^1(T)$ with Teichmüller geodesic τ_q , consider the pullback bundle

$$\mathbb{H}^2 \longrightarrow \mathcal{H}_{\tau_q} \longrightarrow \tau_q.$$

The lifted quadratic differential q_t defines a flat metric on the fiber $\mathcal{H}_{\tau_q(t)} \cong \mathbb{H}^2$. The lifted Teichmüller mapping identifies the fibers $\mathcal{H}_{\tau_q(t)}$ with $\mathcal{H}_{\tau_q(0)}$, determining a homeomorphism $\mathcal{H}_{\tau_q(t)} \cong \tilde{S} \times \mathbb{R}$ so that $(z, 0) \mapsto (z, t)$ is the Teichmüller mapping. The coordinate t and preferred coordinates $\zeta = x + iy$ for q give local coordinates for $S \times \mathbb{R}$ away from $\{\text{singularities of } q\} \times \mathbb{R}$. We thus have a metric $e^{2t}dx^2 + e^{-2t}dy^2 + dt^2$ on $(S - \{\text{singularities of } q\}) \times \mathbb{R}$ whose metric completion is naturally identified with $\tilde{S} \times \mathbb{R} \cong \mathcal{H}_{\tau_q}$, and whose restriction to each fiber is just the metric q_t . We let $\mathcal{H}_{\tau_q}^{\text{SOL}}$ denote \mathcal{H}_{τ_q} with this metric. This is the *singular SOL metric associated to q* .

We now note that Proposition 21 provides an “exponential growth” version of Theorem 17 for the singular SOL metric. Given γ in $\pi_1(S)$, define *isometric* sections $\{\Xi_n\}_{n \in \mathbb{Z}}$ of $\mathcal{H}_{\tau_q}^{\text{SOL}} \rightarrow \tau_q$ by picking linearly ordered points $\{z_n\}_{n \in \mathbb{Z}} = \{(z_n, 0)\}_{n \in \mathbb{Z}} \subset \tilde{\gamma}_0^* \subset \tilde{S} \times \{0\}$. Let $\Xi_n = \{(z_n, t) \mid t \in \mathbb{R}\} \subset \mathcal{H}_{\tau_q}^{\text{SOL}} \cong \tilde{S} \times \mathbb{R}$. By construction, the Ξ_n are linearly ordered over every $\tau_q(t)$. Let δ_0^n denote the segment from z_n to z_{n+1} inside $\tilde{\gamma}_0^*$, so that δ_t^n is the segment from Ξ_n to Ξ_{n+1} inside $\tilde{\gamma}_t^*$. This gives us the following singular SOL variant of Theorem 17.

Proposition 23. *Given $r > 0$, let $T_r > 0$ be as in Proposition 21. Let q be a unit-norm quadratic differential defining an r -thick geodesic τ_q in $\mathcal{T}(S)$ and suppose that γ in $\pi_1(S)$ is nonfilling and balanced at time zero. Given isometric sections $\{\Xi_n\}_{n \in \mathbb{Z}}$ as above with*

$$d_{\tau_q(0)}(\Xi_n, \Xi_{n+1}) = \|\delta_0^n\|_{q_0} \geq e^{T_r},$$

we have

$$d_{\tau_q(t)}(\Xi_n, \Xi_{n+1}) \geq \frac{\varepsilon_0 e^{-T_r}}{4} e^{|t|} d_{\tau_q(0)}(\Xi_n, \Xi_{n+1}). \quad \square$$

Given a unit-norm quadratic differential q defining an r -thick geodesic τ_q in $\mathcal{T}(S)$ and a nonfilling γ in $\pi_1(S)$, the space $\mathcal{A}^{\text{SOL}}(\gamma) = \cup \tilde{\gamma}_t^*$ is δ^{SOL} -hyperbolic for some $\delta^{\text{SOL}} = \delta^{\text{SOL}}(g, r)$. In fact, this space is quasiisometric to the hyperbolic plane. Following the argument (in Section 4.2) that derives Theorem 13 from Theorem 17, we have the following corollary of Proposition 23.

If $[a, b]$ is an interval, we let

$$\mathcal{A}_{[a,b]}^{\text{SOL}} = \bigcup_{a \leq t \leq b} \tilde{\gamma}_t^*.$$

Corollary 24. *Let $r > 0$ and let T_r , q , and γ be as in Proposition 23. There are constants A_0 , K_4 , and C_4 depending only on r and the genus g of S such that the fiber $\tilde{\gamma}_0$ is a (K_4, C_4) -quasigeodesic in $\mathcal{A}^{\text{SOL}}(\gamma)$ and $\mathcal{A}_{[-a,a]}^{\text{SOL}}$ is A_0 -quasiconvex for all a . \square*

Proposition 23 also has the following corollary.

Corollary 25. *Let $R, r > 0$ and let T_r , q , γ , and Ξ_n be as in Proposition 23. There is an $B_2 = B_2(R, r)$ such that if the R -neighborhood of Ξ_n intersects Ξ_m , then $|n - m| \leq B_2$. \square*

We now promote Proposition 23 to a statement about arbitrary (k, c) -sections.

Proposition 26. *Given $r, k, c > 0$, there exists $D_2 > D_3 > 0$ with the following property. Let q be a unit-norm quadratic differential defining an r -thick geodesic τ_q in $\mathcal{T}(S)$ and suppose that γ in $\pi_1(S)$ is nonfilling and balanced at time zero. Suppose that $\{\Sigma_n\}_{n \in \mathbb{Z}}$ are (k, c) -sections contained in $\mathcal{A}^{\text{SOL}}(\gamma) = \cup_t \tilde{\gamma}_t^*$ such that*

$$\{\Sigma_n\}_{n \in \mathbb{Z}} \text{ is linearly ordered over } \tau_q(0) \text{ and } d_{\tau_q(0)}(\Sigma_n, \Sigma_{n+1}) \geq D_3.$$

Then

$$\{\Sigma_n\}_{n \in \mathbb{Z}} \text{ is linearly ordered over } \tau_q(t) \text{ and } d_{\tau_q(t)}(\Sigma_n, \Sigma_{n+1}) \geq D_2 e^{|t|}$$

for every t in \mathbb{R} .

Proof. Let Ξ_n be the isometric sections as in Proposition 23. By Proposition 23, it suffices to show that there is a number B such that if Σ is a (k, c) -section contained in $\mathcal{A}^{\text{SOL}}(\gamma)$, then there are numbers $n > m$ with $n - m \leq B$ such that Σ lies in the region bounded by Ξ_m and Ξ_n .

Let Σ be a (k, c) -section contained in $\mathcal{A}^{\text{SOL}}(\gamma)$. Let $n > m$ be such that Ξ_n and Ξ_m intersect Σ nontrivially.

Pick (z_m, t_m) in $\Xi_m \cap \Sigma$ and (z_n, t_n) in $\Xi_n \cap \Sigma$. Let (w_n, t_m) be the point in $\Xi_n \cap \tilde{\gamma}_{t_m}^*$. Assume that $0 \leq t_m \leq t_n$.

Let $\mathcal{G}_\Sigma: [0, j] \rightarrow \mathcal{A}^{\text{SOL}}$ be a (k, c) -quasigeodesic in Σ joining (z_m, t_m) and (z_n, t_n) . Let \mathcal{G}_Ξ be the geodesic in Ξ_n joining (w_n, t_m) and (z_n, t_n) , let \mathcal{V} be a geodesic in $\mathcal{A}^{\text{SOL}}(\gamma)$ joining (z_m, t_m) and (w_n, t_m) .

By Corollary 24, the set $\mathcal{A}_{[-t_m, t_m]}^{\text{SOL}}$ is A_0 -quasiconvex. So \mathcal{V} lies in A_0 -neighborhood of $\mathcal{A}_{[-t_m, t_m]}^{\text{SOL}}$.

As the space $\mathcal{A}^{\text{SOL}}(\gamma)$ is δ^{SOL} -hyperbolic, it follows that the quasigeodesic triangle $\triangle = \mathcal{G}_\Sigma \cup \mathcal{G}_\Xi \cup \mathcal{V}$ is δ' -thin for some δ' depending only on δ^{SOL} and k and c .

Let $\delta'' = 3 \max\{A_0, \delta'\}$. Since Σ is a (k, c) -section, there is an $i = i(k, c)$ such that

$$\mathcal{G}_\Sigma|_{[i, j]} \subset \mathcal{A}_{[t_m + \delta'', \infty]}^{\text{SOL}}.$$

Since \triangle is δ' -thin and \mathcal{V} is contained in $\mathcal{A}_{[-\infty, t_m + A_0]}^{\text{SOL}}$, the segment $\mathcal{G}_\Sigma|_{[i, j]}$ must lie in the δ' -neighborhood of \mathcal{G}_Ξ . So \mathcal{G}_Σ lies in the $(ki + c + \delta')$ -neighborhood of $\mathcal{G}_\Xi \subset \Xi_n$.

Corollary 25 now bounds $n - m$.

The cases $0 \leq t_n \leq t_m$, $t_m \leq t_n \leq 0$ and $t_n \leq t_m \leq 0$ are proven by essentially the same argument. The cases $t_n \leq 0 \leq t_m$ and $t_m \leq 0 \leq t_n$ are proven by breaking \mathcal{G}_Σ into "positive" and "negative" segments, and running the above argument on each half. \square

The following theorem is due to Farb and Mosher (see Proposition 4.2 of [8] and its proof there), and is the final piece needed to prove Theorem 17.

Theorem 27 (Farb–Mosher [8]). *Given $r, k, c > 0$, there exist K_5, C_5 with the following properties. Suppose $g: \mathbb{R} \rightarrow \mathcal{T}(S)$ is a (k, c) -quasigeodesic that stays a uniformly bounded distance from the r -thick Teichmüller geodesic τ_q and let $v: \mathbb{R} \rightarrow \mathbb{R}$ be a map so that $g(t) \mapsto \tau_q(v(t))$ is the closest point projection. Then this closest point projection is (K_5, C_5) -coarsely Lipschitz and lifts to a fiber-preserving (K_5, C_5) -quasiisometry*

$$\mathcal{H}_g \rightarrow \mathcal{H}_{\tau_q}^{\text{SOL}}$$

for which the maps on fibers $\mathbb{H}_{g(t)} \rightarrow (\tilde{S}, q_{v(t)})$ are (K_5, C_5) -quasiisometries. \square

Proof of Theorem 17. To simplify the discussion, we suppress many of the constants implicit in the proof, and use “uniform” and “uniformly” to mean that the constants involved depend only on the sequence $1 \rightarrow \pi_1(S) \rightarrow \Gamma \rightarrow G \rightarrow 1$ and its associated bundle $\mathbb{H}^2 \rightarrow \mathcal{H}_X \rightarrow X$.

Let Σ_n be our (K_2, C_2) -sections of $\mathcal{H}_X \rightarrow X$.

For every x in X , take a biinfinite geodesic \mathcal{G}_0 in X through x and x_γ . Composing with $X \rightarrow \mathcal{T}(S)$ we get a uniformly quasigeodesic \mathcal{G} fellow travelling an r -thick Teichmüller geodesic τ_q for some $r = r(\Gamma)$. We apply Theorem 27 to produce a uniform fiber-preserving quasiisometry $\mathcal{H}_g \rightarrow \mathcal{H}_{\tau_q}^{\text{SOL}}$. Pushing the $\Sigma_n|_{\mathcal{G}}$ over to $\mathcal{H}_{\tau_q}^{\text{SOL}}$ we obtain uniformly quasiisometric sections Σ'_n . We apply Proposition 26, and push the conclusion back to \mathcal{H}_g . The result is a statement identical to that of Theorem 17 except that x_γ has been replaced with the pullback x_0 of the balance time $\tau_q(0)$. Setting $m'_t = g(t)$ and $\tau_q(v(t))$ (with the appropriate reparameterization) in the discussion at the end of Section 4.4.2, we have $(K_6, C_6) = (K_5, C_5)$, so that (4.12) implies that x_0 is uniformly close to x_γ , and this completes the proof. \square

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